motion in a plane: use of polar coordinates

a- circular motion

A particle moves in a circular path (radius=R) in the xy plane.
Suppose that at time t its cartesian components are given by

$$x(t) = R\cos\theta, \quad y(t) = R\sin\theta$$

where θ is the angular position relative to the x axis measured in the counterclockwise sense. During the motion, in a time Δt the angle changes by $\Delta\theta$, i.e., the new angular position of the particle becomes $\theta + \Delta\theta$; and correspondingly at the later time $t + \Delta t$ the new x coordinate becomes

$$x(t + \Delta t) = R\cos(\theta + \Delta\theta)$$

Therefore, during the time interval Δt, the x coordinate changes by

$$\begin{aligned} \Delta x &= x(t + \Delta t) - x(t) \\ &= R\left\{\cos(\theta + \Delta\theta) - \cos\theta\right\} \end{aligned}$$

For small Δt, and hence for small $\Delta\theta$, one has the approximations

$$\cos\Delta\theta \approx 1, \quad \sin\Delta\theta \approx \Delta\theta$$

Thus, one writes

$$\cos(\theta + \Delta\theta) \approx \cos\theta - \sin\theta\,\Delta\theta$$

and consequently,

$$\Delta x \approx -R\sin\theta\,\Delta\theta$$

Similarly, the variation in the y coordinate can be obtained as given by

$$\Delta y \approx R\cos\theta\,\Delta\theta$$

The displacement vector is therefore

$$\begin{aligned} \Delta\vec{\mathbf{r}} &= \Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}} \\ &= R\left\{-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\right\} \end{aligned}$$

Adopting the convention that the curled fingers of the right hand follow the sense of rotation, one conforms the angular displacement to a vector quantity where the associated direction is that of the extended thumb. Thus, setting

$$\Delta\vec{\theta} = \Delta\theta\,\hat{\mathbf{k}}$$

one sees that the displacement vector can be expressed as

$$\Delta\vec{\mathbf{r}} = \Delta\vec{\theta} \times \vec{\mathbf{r}}$$

Dividing both sides by Δt, one obtains the relation between the translational and rotational (angular) velocities as

$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}, \qquad \vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t}$$

where

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt} = \frac{d\theta}{dt}\,\hat{\mathbf{k}}$$

Evidently, the three vectors $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$ and $\vec{\omega}$ form a triad in the three respective directions $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\mathbf{k}}$, where

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} \\ \hat{\theta} &= -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} \end{aligned}$$

satisfying

1) $\hat{\mathbf{r}} \times \hat{\theta} = \hat{\mathbf{k}}, \qquad \hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\theta}, \qquad \hat{\theta} \times \hat{\mathbf{k}} = \hat{\mathbf{r}}$

2) $\dfrac{d\hat{\mathbf{r}}}{dt} = \hat{\theta}\,\dfrac{d\theta}{dt} = \hat{\theta}\,\omega, \qquad \dfrac{d\hat{\theta}}{dt} = -\hat{\mathbf{r}}\,\dfrac{d\theta}{dt} = -\hat{\mathbf{r}}\,\omega$

i- constant angular velocity

Having obtained the velocity

$$\begin{aligned} \vec{\mathbf{v}} &= \vec{\omega} \times \vec{\mathbf{r}} \\ &= \omega R\,\hat{\theta}, \end{aligned}$$

one can, by straightforward differentiation, derive the acceleration

1) $$\begin{aligned} \vec{\mathbf{a}} &= \frac{d}{dt}(\vec{\omega} \times \vec{\mathbf{r}}) \\ &= \vec{\omega} \times \vec{\mathbf{v}} \\ &= \vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}}) \\ &= (\vec{\omega} \cdot \vec{\mathbf{r}})\,\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\,\vec{\mathbf{r}}) \\ &= -\omega^2\vec{\mathbf{r}} \end{aligned}$$

or

2) $$\vec{\mathbf{a}} = \frac{d}{dt}(\omega R\,\hat{\theta}) = \omega R\,\frac{d\hat{\theta}}{dt} = \omega R\,(-\hat{\mathbf{r}}\,\omega) = -\,\omega^2\vec{\mathbf{r}}$$

In uniform circular motion the acceleration has magnitude: $a = \omega^2 R$, and is directed radially inward toward the centre.